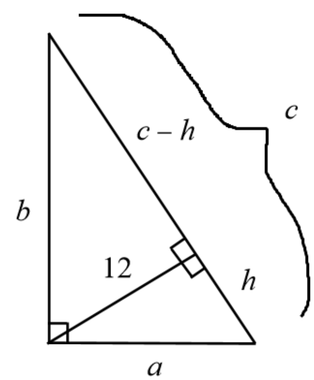
Problem 2

The altitude perpendicular to the hypotenuse of a right triangle is 12 cm. Express the length of the hypotenuse as a function of the perimeter.

Solution

Draw a schematic of the right triangle with the altitude perpendicular to the hypotenuse.



Apply the Pythagorean theorem to the three triangles.

$$\left.\begin{aligned} a^2 + b^2 &= c^2 \\ h^2 + 12^2 &= a^2 \\ (c-h)^2 + 12^2 &= b^2 \end{aligned}\right\}$$

Substitute the second and third equations into the first to get an equation involving only c and h.

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= (h^2 + 12^2) + [(c-h)^2 + 12^2] \\ &= h^2 + 144 + (c^2 - 2ch + h^2) + 144 \\ &= 2h^2 - 2ch + 288 + c^2 \end{aligned}$$

Subtract c^2 from both sides.

$$0 = 2h^2 - 2ch + 288$$

Divide both sides by 2.

$$0 = h^2 - ch + 144$$

Solve for h.

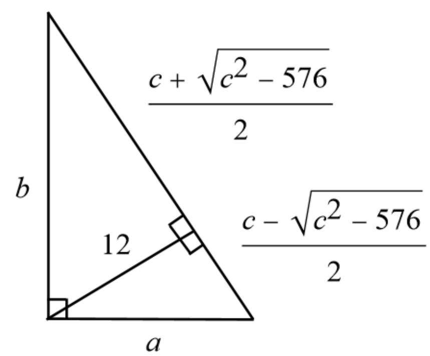
$$h = \frac{c \pm \sqrt{c^2 - 4(144)}}{2} = \left\{ \frac{c - \sqrt{c^2 - 576}}{2}, \frac{c + \sqrt{c^2 - 576}}{2} \right\}$$

Based on the schematic, h is smaller than $c/2$, so

$$h = \frac{c - \sqrt{c^2 - 576}}{2},$$

which means

$$c - h = \frac{c + \sqrt{c^2 - 576}}{2}.$$



With these expressions for h and $c - h$, simplify the formulas for a^2 and b^2.

$$\begin{aligned} a^2 &= h^2 + 12^2 \\ &= \left(\frac{c - \sqrt{c^2 - 576}}{2}\right)^2 + 144 \\ &= \frac{\left(c - \sqrt{c^2 - 576}\right)^2}{4} + 144 \\ &= \frac{c^2 - 2c\sqrt{c^2 - 576} + (c^2 - 576)}{4} + 144 \\ &= \frac{2c^2 - 2c\sqrt{c^2 - 576}}{4} \\ &= \frac{c^2 - c\sqrt{c^2 - 576}}{2} \end{aligned}$$

$$\begin{aligned} b^2 &= (c - h)^2 + 12^2 \\ &= \left(\frac{c + \sqrt{c^2 - 576}}{2}\right)^2 + 144 \\ &= \frac{\left(c + \sqrt{c^2 - 576}\right)^2}{4} + 144 \\ &= \frac{c^2 + 2c\sqrt{c^2 - 576} + (c^2 - 576)}{4} + 144 \\ &= \frac{2c^2 + 2c\sqrt{c^2 - 576}}{4} \\ &= \frac{c^2 + c\sqrt{c^2 - 576}}{2} \end{aligned}$$

The perimeter of the triangle is then

$$\begin{aligned} P &= a + b + c \\ &= \sqrt{h^2 + 12^2} + \sqrt{(c - h)^2 + 12^2} + c \\ &= \sqrt{\frac{c^2 - c\sqrt{c^2 - 576}}{2}} + \sqrt{\frac{c^2 + c\sqrt{c^2 - 576}}{2}} + c. \end{aligned}$$

Use a computer algebra system, such as Mathematica or Maple, to solve this equation for c.

$$c = \frac{P^2}{2(P + 12)}$$